

BASIC RELATIONS FOR THE PERIOD VARIATION MODELS OF VARIABLE STARS

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Abstract: Models of period variations are basic tools for period analyzes of variable stars. We introduce phase function and instant period and formulate basic relations and equations among them. Some simple period models are also presented.

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1. Motivation

The majority of variable stars change brightness (also radial velocity, magnetic field, etc.) more or less regularly and with one period P . As a rule the form and amplitude of their light curves remain constant for plenty of cycles while their periods may change slightly for a number of reasons. The goal of this paper is to create models of such period variations which can then serve as basic tools for the advanced period analysis of changes of variable objects.

2. The instantaneous period and the phase function

The state of a periodically changing variable star is described by two functions of time t : the non-descending stairs-like epoch function $E(t)$ expressing the number of cycles elapsed from the moment beginning of epochs counting M_0 , and the sawtooth function called the phase $\varphi(t)$, reaching its minimum ($\varphi(t) = 0$) when the new cycle of the variability starts and its maximum

($\varphi(t) = 1$) when the particular cycle ends. The phase is used for the construction of so called phase curves of various values characterizing the variable star. The *instantaneous period* for a selected epoch $P(E)$ is then the duration of this epoch. It is useful to introduce a new time-dependent quantity – *phase function* $\vartheta(t)$, as

$$\vartheta(t) = E(t) + \varphi(t); \quad \varphi(t) = \text{frac}[\vartheta(t)]; \quad E(t) = \text{floor}[\vartheta(t)], \quad (1)$$

that could substitute both these ‘mathematically awful’ functions. The operator ‘frac’ removes from a real number its integral part, while ‘floor’ rounds the quantity to the nearest lower integer.

The phase function $\vartheta(t)$ is a monotonic rising function of time originating at the beginning of epoch counting $t = M_0$, $\vartheta(M_0) = 0$. Using $\vartheta(t)$ we are able to determine the epoch and the phase (see Eq. 1) for any time. The time derivative of the phase function at the time t is equal to the instantaneous frequency $\dot{\vartheta}(t) = f(t) = P(t)^{-1}$, where $P(t)$ is the instantaneous period of the variable star at the time t (firstly introduced in [1]). The phase function is then determined by the following differential equation

$$\frac{d\vartheta(t)}{dt} = \frac{1}{P(t)}; \quad \vartheta(t = M_0) = 0; \quad \Rightarrow \quad \vartheta(t) = \int_{M_0}^t \frac{d\tau}{P(\tau)}. \quad (2)$$

It is useful to also introduce the inversion function $T(\vartheta)$ to the phase function ϑ which determines the moment T when the phase function reaches the assigned value. The times of the zero phase (usually the times of brightness minima or maxima) Θ for the distinct epoch E are then given by the relation: $\Theta(E) = T(\vartheta = E)$.

The function $T(\vartheta)$ can be derived as the inversion function of $\vartheta(t)$ or it can be found by solving the differential equation:

$$\frac{dT(\vartheta)}{d\vartheta} = P(\vartheta); \quad T(\vartheta = 0) = M_0; \quad \Rightarrow \quad T(\vartheta) = M_0 + \int_0^\vartheta P(\zeta) d\zeta. \quad (3)$$

3. Some simple period models

Let us derive the phase functions, their inverses and period time dependencies for some astrophysically important polynomial period models.

$$\underline{P(t) = P_0}$$

The simplest period model that we use as a first approximation assumes that the period of variations is constant: $P(t) = P_0$. Then, using the equations (2) and (3), we get for its $\vartheta_1(t)$ and $T_1(\vartheta)$

$$\vartheta_1(t) = \frac{t - M_0}{P_0}; \quad T_1(\vartheta) = M_0 + P_0 \vartheta_1. \quad (4)$$

$$\underline{P(t) = \dot{P}_0}$$

Let's assume that the period is a linear function of the time:

$$\dot{P} = \dot{P}_0; \quad P(t) = P_0 + \dot{P}_0 (t - M_0) = P_0(1 + \dot{P}_0 \vartheta_1), \quad (5)$$

$$\frac{dP}{dt} = \frac{dP}{dT} = \frac{dP}{d\vartheta} \frac{d\vartheta}{dT} = \frac{dP}{d\vartheta} \frac{1}{P} = \dot{P}_0; \quad P(\vartheta) = P_0 e^{\dot{P}_0 \vartheta}; \quad (6)$$

$$\vartheta(t) = \frac{1}{\dot{P}_0} \ln(1 + \dot{P}_0 \vartheta_1); \quad T(\vartheta) = M_0 + \frac{P_0}{\dot{P}_0} (e^{\dot{P}_0 \vartheta} - 1). \quad (7)$$

Since the instantaneous rate of period \dot{P} is as a rule very slow, we can replace the real phase functions $\vartheta(t)$ and their inverses by their Maclaurin decomposition:

$$\vartheta(t) = \frac{1}{\dot{P}_0} \ln(1 + \dot{P}_0 \vartheta_1) \doteq \vartheta_1 - \dot{P}_0 \frac{\vartheta_1^2}{2} + \dot{P}_0^2 \frac{\vartheta_1^3}{3} - \dots; \quad (8)$$

$$T(\vartheta) = M_0 + \frac{P_0}{\dot{P}_0} (e^{\dot{P}_0 \vartheta} - 1) \doteq M_0 + P_0 \left(\vartheta + \dot{P}_0 \frac{\vartheta^2}{2!} + \dot{P}_0^2 \frac{\vartheta^3}{3!} \dots \right). \quad (9)$$

For the majority of variable stars with an inconstant period it is valid that $\dot{P} \Delta t \ll P$, where Δt is the whole duration of its observation. The last decomposition terms in (see Eqs. 8, 9) can then be neglected.

$$\begin{aligned} \vartheta(t) &\doteq \frac{t - M_0}{P_0} - \frac{\dot{P}_0}{2} \left(\frac{t - M_0}{P_0} \right)^2; \quad T(\vartheta) \doteq M_0 + P_0 \vartheta + P_0 \dot{P}_0 \frac{\vartheta^2}{2!}; \\ P(t) &= P_0 + \dot{P}_0 (t - M_0); \quad P(\vartheta) = P_0 (1 + \dot{P}_0 \vartheta). \end{aligned} \quad (10)$$

The phase function and its inverse are quadratic functions and to describe them we just need three parameters: M_0 , P_0 , \dot{P}_0 .

The exact same relations as given in Eqs. (10) can be obtained if we assume other laws for period development, such as $P \dot{P} = P_0 \dot{P}_0$ or $\dot{P}/P = \dot{P}_0/P_0$, or if we assume the parabolic course of the phase function $\vartheta(t)$ or its inverse $T(\vartheta)$, $T(E)$.

$$\underline{\ddot{P} = \ddot{P}_0}$$

If we find that the dependence of $\vartheta(t)$ have apparent cubic or higher terms, we acknowledge that the first derivative of the period is inconstant. Then the simplest period model assumes the period to be a quadratic function of time:

$$P(t) = P_0 + P_0 \dot{P}_0 \vartheta_1 + P_0^2 \ddot{P}_0 \frac{\vartheta_1^2}{2}; \quad \dot{P}(t) = \frac{dP}{d\vartheta_1} \frac{d\vartheta_1}{dt} = \dot{P}_0 + P_0 \ddot{P}_0 \vartheta_1. \quad (11)$$

Using Eq. (2) we can find the exact solution for the phase function $\vartheta(t)$. However, the function expression is so complex that it is practically useless. Consequently we only the first three terms of its McLaurin decomposition.

$$\vartheta(t) \doteq \vartheta_1 - \dot{P}_0 \frac{\vartheta_1^2}{2!} - (P_0 \ddot{P}_0 - 2\dot{P}_0^2) \frac{\vartheta_1^3}{3!} \doteq \vartheta_1 - \dot{P}_0 \frac{\vartheta_1^2}{2!} - P_0 \ddot{P}_0 \frac{\vartheta_1^3}{3!}; \quad (12)$$

$$\begin{aligned} T(\vartheta) &\doteq M_0 + P_0 \vartheta + P_0 \dot{P}_0 \frac{\vartheta^2}{2!} + (P_0^2 \ddot{P}_0 + P_0 \dot{P}_0^2) \frac{\vartheta^3}{3!} \\ &\doteq M_0 + P_0 \vartheta + P_0 \dot{P}_0 \frac{\vartheta^2}{2!} + P_0^2 \ddot{P}_0 \frac{\vartheta^3}{3!}; \end{aligned} \quad (13)$$

$$\frac{P(\vartheta)}{P_0} \doteq 1 + \dot{P}_0 \vartheta + (P_0 \ddot{P}_0 + \dot{P}_0^2) \frac{\vartheta^2}{2!} \doteq 1 + \dot{P}_0 \vartheta + P_0 \ddot{P}_0 \frac{\vartheta^2}{2!}. \quad (14)$$

The expressions in brackets in Eqs. (12), (13), and (14) can be simplified, because of $P_0 \ddot{P}_0 \gg \dot{P}_0^2$ is valid for all known cases of variable stars changing their period.

4. Conclusion

We introduced several simple polynomial period models that can generally be used for most known and unknown period trends. More complicated models of period changes in eclipsing binary systems, accounting for the light time effect caused by third bodies as well as apsidal motion, should also be developed using the relations given in Section 2. However, the formulation of those period models is beyond the scope of this article.

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